Equations in tunable laser optics: brief introduction

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A succinct introduction to the various interference, diffraction, dispersion, and linewidth equations necessary to predict the coherence properties of narrow-linewidth tunable lasers is given. The quantum mechanical origin of the cavity linewidth equation, diffraction, refraction (Snell's law), and reflection, is also referenced.

1. Introduction

The equations included here form the theory that is applied in the design of practical high-power tunable laser oscillators regardless of the type of gain media. In other words, the theory applies for lasers using tunable gain media in the gas, the liquid, or the solid-state. These equations can be used to either predict the value of real measurable variables in laser physics or, alternatively, they can be used to explain the value of measured parameters such as laser beam divergence ($\Delta\theta$) and laser linewidth ($\Delta\nu$). Both these parameters are essential to determine and characterize the coherence of a laser.

The introduction to this subject, given here, is backwards to the chronological development of the physics. But that should not be surprising since, after all, Dirac's notation *is* backwards. The readers should be aware that not all the terms might be explained here and that consultation with the original references is necessary. All these equations are explained in detail, with the necessary geometry and schematics, in *Tunable Laser Optics*.[1]

2. Generalized Dirac interference equations: interferometric quantum origin of the linewidth equation and the uncertainty principle

The generalized probability distribution for propagation from a source s to an interferometric plane x, is given by[1,2]

$$|\langle x|s\rangle|^2 = \sum_{z=1}^{N}\sum_{y=1}^{N}\sum_{x=1}^{N}\Psi(r_{zyx})\sum_{q=1}^{N}\sum_{p=1}^{N}\sum_{r=1}^{N}\Psi(r_{qpr})e^{i(\Omega_{qpr}-\Omega_{zyx})} \tag{1}$$

This equation applies either to the propagation of a single photon or to the propagation of a large number of indistinguishable photons, as in the case of laser radiation.[3,4] In one dimension this equation reduces to[4,5]

$$|\langle x|s\rangle|^2 = \sum_{j=1}^{N}\Psi(r_j)^2 + 2\sum_{j=1}^{N}\Psi(r_j)\left(\sum_{m=j+1}^{N}\Psi(r_m)\cos(\Omega_m - \Omega_j)\right) \tag{2}$$

In this equation it is the cosine term that contains all the information about the geometry and the wavelength of the emission.[1-6] This interference equation has been shown to predict measured interferometric distributions either in the near or the far field.[1-5] Also, it can be used to predict diffraction profiles due to transmission via single-slits. This is done by dividing the single slit into a large number of imaginary sub slits.[1, 2, 4, 5] Further uses of this equation include the derivation of the linewidth cavity equation via quantum principles[1, 6]

$$\Delta\lambda \approx \Delta\theta(\partial\theta/\partial\lambda)^{-1} \tag{3}$$

that can also be expressed as[7]

$$\Delta\lambda \approx \Delta\theta(\nabla_\lambda\theta)^{-1} \tag{4}$$

This equation contains all the essential information necessary to design narrow-linewidth tunable lasers. This equation tells us that the linewidth of a pulsed laser is directly proportional to its beam divergence ($\Delta\theta$) and inversely proportional to the intracavity dispersion $(\nabla_\lambda\theta)$. For the case of a laser cavity including a multiple-prism grating assembly the multiple return-pass linewidth is given by[1, 8]

$$\Delta\lambda = \Delta\theta_R\left(MR\nabla_\lambda\Theta_G + R\nabla_\lambda\Phi_P\right)^{-1} \tag{5}$$

where M is the intracavity beam expansion, *R* is the total number of return passes to the onset of laser emission, $\nabla_{\lambda}\Theta_G$ is the dispersion of the grating, $\nabla_{\lambda}\Phi_P$ is the double-pass prismatic dispersion, and[9]

$$\Delta\theta_R = (\lambda / \pi w)\left(1 + (L_\mathrm{R} / B_R)^2 + (A_R L_\mathrm{R} / B_R)^2\right)^{1/2} \tag{6}$$

is the generalized expression for the beam divergence which is a function of the Rayleigh length and propagation matrix terms.

Equation (4) also explains the physics behind femtosecond, or ultrafast, lasers. In that case the intracavity dispersion is reduced to a minimum thus allowing for broadband emission and hence, via the uncertainty principle, to ultrashort pulse emission.[1] At this stage it is appropriate to indicate that the interferometric equation (Eq. (2)) can be used to yield an approximate derivation[1] of Heisenberg's uncertainty principle[10-12]

$$\Delta x \Delta p \approx h \tag{7}$$

which, in turn, can be used to derive an expression for the diffraction limit of beam divergence[1, 13]

$$\Delta\theta \approx \lambda / \pi w \tag{8}$$

where *w* is known as the beam waist. This is the minimum expression for beam divergence under ideal conditions. In practical lasers, as depicted in Eq. (6), this expression is multiplied by the square root of a series of terms derived from the geometry of the resonator[1] Under ideal circumstances that term reduces to ~ 1.

Thus we have described, very succinctly , how generalized interferometic equations derived using the Dirac notation can be used to yield all the fundamental concepts necessary to design narrow-linewidth tunable laser oscillators and femtosecond, or ultrafast, lasers . Further, this approach has been used to provide a unified quantum description of optics in the following order: interference, diffraction, refraction (Snell's law), and reflection.[1, 3, 5] Next we consider the dispersion term in the linewidth equation.

3. Generalized multiple-prism grating dispersion equations

The subject of multiple-prism dispersion was first discussed, in a qualitative manner, by Newton in his prophetic book *Opticks*.[14] Subsequently, Brewster described the use of prism pairs.[15] However, a generalized mathematical description of multiple-prism beam expanders and their dispersion was only made available following the event of the tunable laser.[16] Detailed reviews on this subject are given elsewhere.[17-20] Also, the quantum origin of refraction has been described elsewhere.[5, 21] Briefly, using the law refraction (Snell's law) as a staring point, for a multiple-prism array, the cumulative *single-pass* dispersion at the *m*th prism is given by [1, 16-20]

$$\nabla_\lambda \phi_{2,m} = \mathcal{H}_{2,m} \nabla_\lambda n_m + (k_{1,m} k_{2,m})^{-1} \left(\mathcal{H}_{1,m} \nabla_\lambda n_m \pm \nabla_\lambda \phi_{2,(m-1)} \right) \tag{9}$$

this is a recursive function that depends on the dispersion of the previous prism ($m-1$) and where the $k_{1,m}$ term, for instance, refers to the beam expansion undergone by the light beam following incidence on the first surface of the *m*th prism. Also, n_m is the refractive index of the *m*th prism and $\mathcal{H}_{1,m}$, $\mathcal{H}_{2,m}$ are additional geometrical terms defined in the references. An explicit equation for the *double-pass* intracavity dispersion, including terms describing the overall beam expansion M_1 and M_2, is given by[19]

$$\nabla_\lambda \Phi_P = 2 M_1 M_2 \sum_{m=1}^{r} (\pm 1) \mathcal{H}_{1,m} \left(\prod_{j=m}^{r} k_{1,j} \prod_{j=m}^{r} k_{2,j} \right)^{-1} \nabla_\lambda n_m$$

$$+ 2 \sum_{m=1}^{r} (\pm 1) \mathcal{H}_{2,m} \left(\prod_{j=1}^{m} k_{1,j} \prod_{j=1}^{m} k_{2,j} \right) \nabla_\lambda n_m \tag{10}$$

for identical prisms deployed at Brewster's angle of incidence this equation reduces to the succinct expression[19]

$$\nabla_\lambda \Phi_P = 2 \sum_{m=1}^{r} (\pm 1) (n_m)^{m-1} \nabla_\lambda n_m \tag{11}$$

Now, going back to the issue of pulse compression: a proper discussion of this phenomenon requires a mathematical description of both the first order dispersion $\nabla_n \phi_{2,m}$ and the second order dispersion[1, 22]

$$\nabla_n^2 \phi_{2,m} = \mathcal{H}_{2,m} (\nabla_n \phi_{2,m})^2 n_m + (k_{1,m} k_{2,m})^{-1}$$

$$\times \{ (\mathcal{H}_{1,m} \chi_{1,m} k_{1,m} \nabla_n \phi_{1,m} n_m \pm \nabla_n^2 \phi_{2,(m-1)}) + (\mathcal{H}_{1,m} \pm \nabla_n \phi_{2(m-1)})$$

$$(\chi_{1,m} \nabla_n \psi_{1,m} - \mathcal{H}_{2,m} k_{2,m} \nabla_n \psi_{2,m} - \chi_{1,m} k_{1,m} \nabla_n \phi_{1,m} n_m) \}$$

$$+ (k_{2,m})^{-1} (\nabla_n \psi_{1,m} + \nabla_n \psi_{2,m}) \tag{12}$$

Further, it should be mentioned that multiple-prism arrays were first described in matrix form in 1989.[23] It should also be mentioned that the generalized beam expansion coefficient and dispersion, can be integrated as components of 4×4 propagation matrices as described elsewhere.[1, 24]

In order to include the case of negative refraction in the generalized dispersion description Eq. (9) takes the more general form of[21]

$$\nabla_\lambda \phi_{2,m} = \pm \mathcal{H}_{2,m} \nabla_\lambda n_m \pm (k_{1,m} k_{2,m})^{-1} \left(\mathcal{H}_{1,m} \nabla_\lambda n_m (\pm) \nabla_\lambda \phi_{2,(m-1)} \right) \tag{13}$$

where the signs in parenthesis refer to the geometrical configuration whilst the simple ± refers to either positive (+) or negative (–) refraction.

Albeit substantial progress had been made towards the mathematical/theoretical representation of the generalized multiple-prism dispersion[16, 22] complete access to higher phase derivatives has only recently been granted For instance, the 5th derivative of the generalized multiple-prism refraction, or the 4th derivative of the generalized multiple-prism dispersion, is elegantly given by[25]

$$\begin{aligned}\nabla_n^5\phi_{2,m} &= \nabla_n^4\mathcal{H}_{2,m} \\ &+ (\nabla_n^4\mathcal{M}^{-1})(\mathcal{H}_{1,m} \pm \nabla_n\phi_{2,(m-1)}) \\ &+ 4(\nabla_n^3\mathcal{M}^{-1})(\nabla_n\mathcal{H}_{1,m} \pm \nabla_n^2\phi_{2,(m-1)}) \\ &+ 6(\nabla_n^2\mathcal{M}^{-1})(\nabla_n^2\mathcal{H}_{1,m} \pm \nabla_n^3\phi_{2,(m-1)}) \\ &+ 4(\nabla_n\mathcal{M}^{-1})(\nabla_n^3\mathcal{H}_{1,m} \pm \nabla_n^4\phi_{2,(m-1)}) \\ &+ (\mathcal{M}^{-1})(\nabla_n^4\mathcal{H}_{1,m} \pm \nabla_n^5\phi_{2,(m-1)})\end{aligned} \tag{14}$$

In this most current extension to the generalized multiple-prism dispersion theory a clear and elegant mathematical framework is provided to express, at will, any higher derivatives up to the *N*th order.[25] In reference to Eq. (14), for instance, the keen observer can recognize , from the second to the fifth term, the elements of Pascal's triangle for the power of 4.

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